Massless particles of any spin obey linear equations of motion

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Abstract

The proof is presented that the Poincaré symmetry determines the equations of motion for massless particles of any spin in $2n$-dimensional spaces, which are linear in the momentum [1]: $(W^a = \alpha p^a)|\Phi\rangle$, with W^a the generalized Pauli-Ljubanski vector.

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1 Introduction

All theories with $d > 4$ have to answer the question why Nature has made a choice of four-dimensional subspace with one time and three space coordinates and with the particular choice of charges beside the spin degree of freedom for either fermions or bosons. One of us and Nielsen [1,2] has prooved that in d-dimensional spaces, with even d, the spin degrees of freedom require q time

[1] After this paper appeared on hep-th W. Siegel let us know that he proposed[13,14] the equations $(S^{ab}p_b + wp^a = 0)|\Phi\rangle$ which are linear in p^a-momentum, as well as in S^{ab} for all irreducible representations of massless fields in any d. Following derivations of this paper one easily proves that solutions of Siegel's equations belong to irreducible representations of the Poincaré group. The proof is much simpler than for our equations $(W^a = \alpha p^a)|\Phi\rangle$. Both equations are of course equivalent. Following our derivations one finds that the constant w in the Siegel's equation is $w = l_n$, with l_n defined in this paper. One derives our equation from the Siegel's one for even d after some rather tedious calculations if multiplying it by $\varepsilon_{aca_1a_2\ldots a_{d-3}a_{d-2}}S^{a_1a_2}\cdots S^{a_{d-3}a_{d-2}}$.

and $(d-q)$ space dimensions, with q which has to be odd. Accordingly in four-dimensional space Nature could only make a choice of the Minkowsky metric. This proof was made under the assumption that equations of motion are for massless fields of any spin linear in the d-momentum $p^a,\ a=0,1,2,3,5,\dots,d$. (In addition, also the Hermiticity of the equation of motion operator as well as that this operator operates within an irreducible representation of the Lorentz group was required.) Our experiences tell us that equations of motion of all known massless fields are linear in the four-momentum $p^a,\ a=0,1,2,3$. We are refering to the Dirac equation of motion for massless spinor fields and the Maxwell or Maxwell-like equations of motion for massless bosonic fields. One of us together with A. Borštnik [3–6] has shown that the Weyl-like equations exist not only for fermions but also for bosons. For four dimensional space-time Wigner [7] classified the representations of the Poincaré group, connecting them with particles. The classification of representations can also be found in Weinberg [8], for example. According to these classifications, equations of motion follow, when constraining a solution space to a certain Poincaré group representation. For spinors this leads to the Dirac equation and for vectors to the Maxwell equations [9].

The aim of this paper is to briefly present the proof (the detailed version is presented in Ref. [10]) that in even dimensional spaces, for any $d=2n$, free massless fields $|\Phi\rangle$ ($(p^a p_a = 0)\ |\Phi\rangle,\quad a=0,1,2,3,5,\dots,2n$) of any spin satisfy equations of motion which are linear in the d-momentum $p^a=(p^0,\overrightarrow{p})$

$$
\begin{aligned}
&(W^a = \alpha p^a)\ |\Phi\rangle, \quad a=0,1,2,3,5,\dots,d,\\
\text{with}\quad &\alpha = -\rho 2^{n-1}(n-1)!(l_{n-1}+n-2)\dots(l_2+1)l_1
\end{aligned} \tag{1}
$$

and guarantee the validity of the equation $(p^a p_a = 0)\ |\Phi\rangle$.

In Eq. (1) the operators S^{ab} are the generators of the Lorentz group $SO(1,d-1)$ in internal space, which is the space of spin degrees of freedom. Parameters $l_1,\dots,l_n$ are eigenvalues of the operators of the Cartan (Eq.(9)) subalgebra of the algebra of the group $SO(1,d-1)$ on the maximal weight state (13) of an irreducible representation. Vector W^a is the generalized Pauli-Ljubanski [4] d-vector

$$
W^a := \rho\,\varepsilon^{ab}{}_{a_1a_2\dots a_{d-3}a_{d-2}}p_b S^{a_1a_2}\dots S^{a_{d-3}a_{d-2}}. \tag{2}
$$

The value of ρ is irrelevant in the equations of motion (1) since it cancels out. It becomes relevant with the introduction of $d-1$ vector [1,2]

$$
S^i := \rho\,\varepsilon^{0i}{}_{a_1a_2\dots a_{d-3}a_{d-2}}S^{a_1a_2}\dots S^{a_{d-3}a_{d-2}}. \tag{3}
$$

We choose the value of ρ in such a way that S^i has eigenvalues independent of

dimension d and with values familiar from four dimensions. For spinors, which are determined by Eqs.(8) or equivalently by $1/2 = l_n = \ldots = l_2 = \pm l_1$ we have

$$\alpha = \pm\frac{1}{2}, \quad \rho = \frac{2^{n-2}}{(2n-2)!} \tag{4}$$

and for vector fields with $1 = l_n = \ldots = l_2 = \pm l_1$ we have

$$\alpha = \pm 1, \quad \rho = \frac{1}{2^{n-1}(n-1)!^2}. \tag{5}$$

The proof is made only for fields with no gauge symmetry and with a nonzero value of the *handedness* operator [4,5] $\Gamma^{(int)}$

$$\begin{aligned} \Gamma^{(int)} &:= \beta\, \varepsilon_{a_1 a_2 \ldots a_{d-1} a_d} S^{a_1 a_2} \ldots S^{a_{d-1} a_d}, \\ \beta &= \frac{i}{2^n n!(l_n + n - 1)\ldots(l_2+1) l_1}, \quad d = 2n, \end{aligned} \tag{6}$$

which commutes with all the generators of the Poincaré group. We choose β so that $\Gamma^{(int)} = \pm 1$ on representations with nonzero handedness. For spinors (Eq.(8)) $\beta = (2^n i)/(2n)!$, while for vector fields it is $\beta = i/(2^n (n!)^2)$.

We prove that for spinors in d-dimensional space Eq.(1) is equivalent to the equation

$$(\Gamma^{(int)} p^0 = \frac{1}{|\alpha|}\ \overrightarrow{S} \cdot \overrightarrow{p})|\Phi\rangle. \tag{7}$$

with $1/|\alpha|$ equal to 2 for any $d = 2n$ while for a general spin Eq.(1) may impose additional conditions on the field.

We recognize the generators S^{ab} to be of the spinorial character, if they fulfil the relation

$$\{S^{ab}, S^{ac}\} = \frac{1}{2}\, \eta^{aa}\, \eta^{bc}, \ \text{ no summation over a}, \tag{8}$$

with $\{A, B\} = AB + BA$.

In this paper the metric is, independantly of the dimension, assumed to be the Minkowsky metric with $\eta^{ab} = \delta^{ab}(-1)^A,\ A = 0$ for $a = 0$ and $A = 1$, otherwise.

We present the proof in steps introducing only the very needed quantities and assuming that the reader can find the rest in text-books, as well as in Ref. [10].

2 Irreducible representations of the Lorentz group

We denote an irreducible representation of the Lorentz group $SO(1,d-1)$ by the weight of the dominant weight state of the representation [11].

The Lie algebra of $SO(1,d-1)$ is spanned by the generators S^{ab}, which satisfy the commutation relations $[S^{ab},S^{cd}] = i(\eta^{ad}\, S^{bc} + \eta^{bc}\, S^{ad} - \eta^{ac}\, S^{bd} - \eta^{bd}\, S^{ac})$. We choose the n commuting operators of the Lorentz group $SO(1,d-1)$ as follows

$$- iS^{0d}, S^{12}, S^{35}, \ldots, S^{d-2\; d-1} \tag{9}$$

and call them $C_0, C_1, C_2, \ldots, C_{n-1}$ respectively. We say that a state $|\Phi_w\rangle$ has the weight $(w_0, w_1, w_2, \ldots, w_{n-1})$ if the following equations hold

$$C_j|\Phi_w\rangle = w_j|\Phi_w\rangle, \quad j = 0, 1, \ldots, n-1. \tag{10}$$

According to the definition of the operators (Eq.(9)), weight components $w_0, w_1, w_2, \ldots, w_{n-1}$ are always real numbers.

We introduce in a standard way [12] the raising and the lowering operators

$$E_{jk}(\lambda,\mu) := \frac{1}{2}((-i)^{\delta_{j0}} S^{j_- k_-} + i\lambda S^{j_+ k_-} - i^{1+\delta_{j0}}\mu S^{j_- k_+} - \lambda\mu S^{j_+ k_+}), \tag{11}$$

with $0 \leq j < k \leq n-1$, $\lambda,\mu = \pm 1$ and $0_- = 0$, $0_+ = d$, $1_- = 1$, $1_+ = 2$, $2_- = 3$, $2_+ = 5$ and so on. Due to the commutation relations

$$[E_{jk}(\lambda,\mu), C_l] = (\delta_{jl}\lambda + \delta_{kl}\mu)E_{jk}(\lambda,\mu), \tag{12}$$

if the state $|\Phi_w\rangle$ has the weight $(w_0, w_1, \ldots, w_{n-1})$ then the state $E_{jk}(\lambda,\mu)|\Phi_w\rangle$ has the weight $(\ldots, w_j+\lambda, \ldots, w_k+\mu, \ldots)$. We call the state $|\Phi_l\rangle$ with the property

$$E_{jk}(+1,\pm 1)|\Phi_l\rangle = 0, \quad 0 \leq j < k \leq n-1, \tag{13}$$

the dominant weight state. All the other states of an irreducible representation are obtained by the application of the generators S^{ab}. We shall denote an

irreducible representation of the Lorentz group $SO(1,d-1)$ by the weight of the dominant weight state: $(l_n, l_{n-1}, \ldots, l_2, l_1)$. Numbers $l_n, l_{n-1}, \ldots, l_2, l_1$ are either all integer or all half integer and satisfy $l_n \geq l_{n-1} \geq \ldots \geq l_2 \geq |l_1|$.

We can write Eq.(13) in an equivalent way

$$\begin{aligned}(S^{0i} + S^{id})|\Phi_l\rangle &= 0, \quad i = 1,2,3,5,\ldots,d-1\\ (S^{1i} + iS^{2i})|\Phi_l\rangle &= 0, \quad i = 3,5,\ldots,d-1,\\ (S^{3i} + iS^{5i})|\Phi_l\rangle &= 0, \quad i = 6,7\ldots,d-1, \text{ and so on.}\end{aligned} \tag{14}$$

By applying $\Gamma^{(int)} = \beta\varepsilon_{a_1a_2\ldots a_{d-1}a_d} S^{a_1a_2}S^{a_3a_4}\ldots$ (Eq.(6)) on a state with the dominant weight and taking into account Eqs.(14), we find

$$(\Gamma^{(int)} = 2^n n! i\beta(l_n + n - 1)(l_{n-1} + n - 2)\ldots(l_2 + 1)l_1)|\Phi_l\rangle. \tag{15}$$

In order to obtain $\Gamma^{(int)} = \pm 1$ for any irreducible representation in any $d = 2n$, β must be the one, presented in Eq.(6).

3 The unitary discrete massless representations of the Poincaré group

The generators of the Poincaré group, that is the generators of translations p^a and the generators of the Lorentz transformations M^{ab} (which form the Lorentz group), fulfil in any dimension d, even or odd, the commutation relations:

$$\begin{aligned}[p^a, p^b] &= 0,\\ [M^{ab}, M^{cd}] &= i(\eta^{ad}\, M^{bc} + \eta^{bc}\, M^{ad} - \eta^{ac}\, M^{bd} - \eta^{bd}\, M^{ac}),\\ [M^{ab}, p^c] &= i(\eta^{bc}\, p^a - \eta^{ac}\, p^b).\end{aligned}$$

For momenta p^a appearing in an irreducible massless representations of the Poincaré group it holds $p^a p_a = 0, \quad p^0 > 0$ or $p^0 < 0$ (we omit the trivial case $p^0 = 0$). We denote $r = p^0/|p^0|$. The Poincaré group representations are then characterized by the representation of the *little group*, which is a subgroup of the Lorentz group leaving some fixed d-momentum $p^a = \mathbf{k}^a$, satisfying equation $\mathbf{k}^a\mathbf{k}_a = 0$, unchanged. Making the choice of $\mathbf{k}^a = (rk^0, 0, \ldots, 0, k^0)$, with $r = \pm 1, k^0 > 0$, the infinitesimal generators of the little group $\omega_{bc}M^{bc}$, ($\omega_{bc} = -\omega_{cb}$) can be found by requiring that, when operating on the d-vector $\mathbf{k}^a$ give zero, so that accordingly the corresponding group transformations leave the

d-vector $\mathbf{k}^a$ unchanged: $(\omega_{bc}M^{bc})\mathbf{k}^a = 0$. It is easily checked that this requirement leads to equations $\omega_{0d} = 0, \;\; \omega_{i0} + r\omega_{id} = 0, \quad i = 1, 2, 3, 5, \ldots d-1$.

All $\omega_{bc}M^{bc}$ with ω_{bc} subject to conditions $(\omega_{bc}M^{bc})\mathbf{k}^a = 0$ form the Lie algebra of the little group. We choose the following basis of the little group Lie algebra

$$\begin{aligned} \Pi_i = M^{0i} + rM^{id}, \quad & i = 1, 2, 3, 5, \ldots, d-1 \\ \text{and all } \; M^{ij}, \quad & i, j = 1, 2, 3, 5, \ldots, d-1. \end{aligned} \tag{16}$$

One finds

$$[\Pi_i, \Pi_j] = 0, \quad [\Pi_i, M^{jk}] = i(\eta^{ij}\Pi_k - \eta^{ik}\Pi_j). \tag{17}$$

We are interested only in unitary discrete representations of the Poincaré group. This means that the states in the representation space can be labeled by the momentum and an additional label for internal degrees of freedom, which can only have *discrete* values.

Lemma 1: For a discrete representation of the Poincaré group operators Π_i give zero $\Pi_i|\Phi_a\rangle = 0$.

Proof: We may arrange the representation space of the little group so that the commuting operators $\Pi_1, \ldots, \Pi_{d-1}$ are diagonal: $\Pi_i|\Phi_a\rangle = b^i_a|\Phi_a\rangle$. Making the rotation $e^{i\theta M^{ij}}|\Phi_a\rangle$ for a continious set of values of the parameter θ and for the chosen indices i, j we find that the states $e^{i\theta M^{ij}}|\Phi_a\rangle$ can only have the discrete eigenvalues for Π_i if $b^i_a = 0$ for all a and i. (The detailed version of the proof can be found in Ref. [10]). ■

We now introduce the decomposition of the generators of Lorentz transformations to external and internal space. We write $M^{ab} = L^{ab} + S^{ab}$ where $L^{ab} = x^a p^b - x^b p^a$ and S^{ab} are the generators of the Lorentz transformations in internal space (ie. spin generators). We see that on the representation space of the little group with the choice $\mathbf{k}^a = (rk^0, 0, \ldots, 0, k^0)$, on which $(L^{0i} + rL^{id})|\Phi_a\rangle = 0$, the following holds:

$$\Pi_i|\Phi_a\rangle = \Pi_i^{(int)}|\Phi_a\rangle = 0, \quad \Pi_i^{(int)} := (S^{0i} + rS^{id}), \quad \text{for each } \; i. \tag{18}$$

The only little group generators, which are not necesserily zero on the representation space, are $M^{ij}, i, j = 1, 2, \ldots, d-1$ and they form the Lie algebra of $SO(d-2)$. Since the irreducible discrete representations of the Poincaré group in $d(=2n)$ dimensions for massless particles are determined by the irreducible representations of the group $SO(d-2)$ we will denote the former with the same symbol as the latter with an additional label r (ie. energy sign): $(l_{n-1}, l_{n-2}, \ldots, l_2, l_1; r)$.

Lemma 2: On the representation space of an irreducible massless representation of the Poincaré group $(l_{n-1}, \ldots, l_1; r)$ the equation (1) holds with $\alpha = -\rho r 2^{n-1}(n-1)!(l_{n-1}+n-2)\ldots(l_2+1)l_1$.

Proof: First, we prove the lemma on the representation space of the little group for the choice $p^a = \mathbf{k}^a = (rk^0, 0, \ldots, 0, k^0),\ k^0 > 0, r = \pm 1$. We begin with the cases $a = 1, 2, \ldots, d-1$ in Eq.(1)

$$\begin{aligned} W^a/\rho|\Phi_a\rangle = (\varepsilon^{aa_1}{}_{a_2a_3\ldots a_{d-1}} p_{a_1} M^{a_2a_3} M^{a_4a_5}\ldots)|\Phi_a\rangle = \\ -2k^0(n-1)\boldsymbol{\alpha}^{ai}\Pi_i|\Phi_a\rangle, \end{aligned} \tag{19}$$

where

$$\boldsymbol{\alpha}^{ai} := \varepsilon^{0dai}{}_{a_1a_2\ldots a_{d-4}} M^{a_1a_2}\ldots M^{a_{d-5}a_{d-4}}, \quad [\Pi^i, \boldsymbol{\alpha}^{ai}] = 0. \tag{20}$$

Taking into account Eq.(18) we conclude that for $a = 1, 2, \ldots, d-1$, $W^a|\Phi_a\rangle = p^a|\Phi_a\rangle = 0$.

For $a = 0$ and $a = d$ we obtain

$$\begin{aligned} W^0/\rho|\Phi_a\rangle &= (-r)p^0\Gamma^{(int)}_{d-2}/\beta|\Phi_a\rangle, \\ W^d/\rho|\Phi_a\rangle &= (-r)p^d\Gamma^{(int)}_{d-2}/\beta|\Phi_a\rangle, \end{aligned} \tag{21}$$

where $\Gamma^{(int)}_{d-2}$ is the handedness operator corresponding to the subgroup $SO(d-2) \le SO(1, d-1)$ acting on coordinates $1, \ldots, d-1$. Substituting for $\Gamma^{(int)}_{d-2}$ the appropriate value (Eq.(15) with $n-1$ instead of n and without the i factor) we conclude the proof for the choice $p^a = \mathbf{k}^a = (rk^0, 0, \ldots, 0, k^0)$. To extend the proof for Eq.(1) to the whole representation space one only has to note that Eq.(1) is in a covariant form and must therefore hold generally. ■

Since Eq.(1) with α from Eq.(1) holds on the Poincaré group representation $(l_{n-1}, \ldots, l_1; +1)$ it is a candidate for an equation of motion.

4 Equations of motion for free massless fields of any spin in $d = 2n$

What we have to prove is that the solutions of Eq.(1) belong to an irreducible unitary discrete representation, possibly with a degeneracy in the energy sign.

In all proofs that follow we shall make the choice $p^a = \mathbf{k}^a = (rk^0, 0, \ldots, 0, k^0)$, $k^0 > 0, r = \pm 1$, since then the covariance of Eq.(1) guarantees that the proofs are valid for general p^a.

Lemma 3: On the space with internal Lorentz group $SO(1, d-1)$ representation $(l_n, l_{n-1}, \dots, l_2, l_1)$, states satisfying the discreteness condition of Eq.(18) for the little group, form the $SO(d-2)$-irreducible representation space $(l_{n-1}, \dots, rl_1)$, where the subgroup $SO(d-2) \leq SO(1, d-1)$ acts on coordinates $1, 2, \dots, d-1$.

Proof: We present an outline of the proof for $r = 1$ only. The case $r = -1$ is treated similarly (see [10]).

Eq.(17) implies that the space of solutions $\{|\Phi_a\rangle\}$ of Eq.(18) forms an $SO(d-2)$-invariant space, where $SO(d-2) \leq SO(1, d-1)$ acts on coordinates $1, 2, \dots, d-1$. To show that $\{|\Phi_a\rangle\}$ is irreducible and corresponds to the $SO(d-2)$-representation $(l_{n-1}, \dots, l_1)$, we choose any state $|\Phi_a\rangle \in \{|\Phi_a\rangle\}$ with a $SO(d-2)$-dominant weight and prove both that it is unique up to a scalar multiple and has $SO(d-2)$-weight $(l_{n-1}, \dots, l_1)$. The set of states $\{|\Phi_a\rangle\}$ is nontrivial, because the $SO(1, d-1)$-dominant weight state is in $\{|\Phi_a\rangle\}$.

Since $|\Phi_a\rangle$ has a $SO(d-2)$-dominant weight the last two lines in Eqs.(14) must hold. Eq.(18) then implies that the first line of Eq.(14) must also hold. It follows then that the state $|\Phi_a\rangle$ has a $SO(1, d-1)$-dominant weight and is (up to a scalar multiple) unique and the proof that the corresponding representation of the Poincaré group is determined by all but the first dominant weight component of the internal Lorentz group representation $(l_{n-1}, l_{n-2}, \dots, l_2, l_1)$ is complete. ■

It remains to answer the question: when are the solutions of equations $(W^a = \alpha p^a)|\Phi\rangle$, with $\alpha = -\rho 2^{n-1}(n-1)!(l_{n-1} + (n-2)) \dots (l_2 + 1)l_1$ on the space with internal Lorentz group $SO(1, d-1)$ representation $(l_n, l_{n-1}, \dots, l_2, l_1)$, exactly those described by the lemma 3, ie. $(l_{n-1}, \dots, rl_1; r), r = \pm 1$.

Lemma 4: On the space with internal Lorentz group $SO(1, d-1)$ representation $(l_n, l_{n-1}, \dots, l_1)$, where $l_1 \neq 0 \iff \Gamma^{(int)} \neq 0$, the solutions of Eq.(1) are exactly $(l_{n-1}, \dots, rl_1; r)$, with $r = \pm 1$.

Proof: First, we take the simplest case of $d = 4$, proving that on the space with internal Lorentz group $SO(1, 3)$ representation (l_2, l_1) the solutions of Eq.(1) are exactly $(l_1; +1)$ and $(-l_1; -1)$.

In this case, the first equation of Eqs.(1), with $a = 1, 2$ reads $(\Pi_2^{(int)} = 0)|\Phi\rangle$, $(\Pi_1^{(int)} = 0)|\Phi\rangle$, which are exactly Eq.(18) and lemma 3 completes the proof since cases $a = 0, 3$ both give the equation $S^{12}|\Phi\rangle = rl_1|\Phi\rangle$ which imposes no additional constraints on representation spaces $(l_1; +1)$ and $(-l_1; -1)$.

We can proceed now with the general case $d \geq 6$. ¿From Eqs.(19), (21) we know that Eq.(1) can be written for our choice of p^a as follows

$$\boldsymbol{\alpha}^{ij}\Pi_j^{(int)}|\Phi\rangle = 0, \quad \text{for } \; a = i,$$
$$-\rho\varepsilon^{0d}{}_{a_1 a_2 \ldots a_{d-3} a_{d-2}} S^{a_1 a_2} S^{a_3 a_4} \ldots |\Phi\rangle = \alpha|\Phi\rangle, \quad \text{for } \; a = 0 \; \text{ and } \; a = d, \tag{22}$$

with $\boldsymbol{\alpha}^{ij}$ defined in Eq.(20).

By lemma 3 it is sufficient to show that every solution $|\Phi\rangle$ of Eqs.(22) satisfies the condition

$$\Pi_i^{(int)}|\Phi\rangle = 0, \quad \text{for } i = 1, 2, \ldots, d-1. \tag{23}$$

According to the proof of lemma 3 the dominant weight of the group $SO(1, d-1)$ satisfies all the above equations and we can conclude that the space V of solutions of equations in Eq.(22) is nontrivial. It is also $SO(d-2)$-invariant due to the commutation relations $[\boldsymbol{\alpha}^{ij}\Pi_j^{(int)}, S^{kl}] = i(\eta^{ik}\boldsymbol{\alpha}^{lj}\Pi_j^{(int)} - \eta^{il}\boldsymbol{\alpha}^{kj}\Pi_j^{(int)})$ and $[\varepsilon^{0d}{}_{a_1 a_2 \ldots a_{d-3} a_{d-2}} S^{a_1 a_2} \ldots S^{a_{d-3} a_{d-2}}, S^{kl}] = 0$ for $i, j, k, l = 1, 2, \ldots, d-1$.

To prove that Eq.(23) holds on V it suffices to prove that Eq.(23) holds on any $SO(d-2)$-dominant weight state in V. This is done with the aid of Eqs.(22), (14) and some lengthy but elementary calculations (details can be found in [10]).

■

We may now write down the main result of this letter.

On the space with internal Lorentz group representation $(l_n, l_{n-1}, \ldots, l_1)$ *where* $l_1 \neq 0$ *equations*

$$(W^a = \alpha p^a)|\Phi\rangle, \alpha = -\rho 2^{n-1}(n-1)!(l_{n-1} + n - 2)\ldots(l_2 + 1)l_1 \tag{24}$$

are equations of motion for massless particles corresponding to the following representations of the Poincaré group

$$(l_{n-1}, \ldots, l_1; +1) \quad \text{and} \quad (l_{n-1}, \ldots, -l_1; -1), \tag{25}$$

where the masslessness condition $(p_a p^a = 0)|\Phi\rangle$ *is not needed, since it follows from (24) and they are linear in the* p^a*-momentum.*

With the aid of Eqs. (15), (6) the equation of motion can also be written as

$$(W^a = |\alpha|\Gamma^{(int)} p^a)|\Phi\rangle,$$
$$|\alpha| = \rho 2^{n-1}(n-1)!(l_{n-1} + n - 2)\ldots(l_2 + 1)|l_1|. \tag{26}$$

This equation is convenient when dealing with positive and negative handedness on the same footing (an example of this is the Dirac equation) since $|\alpha|$ is independent of the sign of l_1.

We note that the particular value of ρ is irrelevant in Eqs. (24), (26) since ρ is found in both the lefthand and the righthand side of equations and thus cancels out. (The value of ρ becomes relevant, as we have already said, when dealing with the particular spin when it is used to insure that the operators S^i have the familiar values independent of the dimension.)

Making a choice of $a = 0$ one finds

$$(\overrightarrow{S}.\overrightarrow{p} = |\alpha|\Gamma^{(int)}p^0)|\Phi\rangle. \tag{27}$$

Since $(\Gamma^{(int)})^2 = 1$, one immediately finds that $|\alpha| = |\overrightarrow{S}.\overrightarrow{p}|/|p^0|$. The rest of equations make no additional requirements for spinors, while this is not the case for other spins.

5 Concluding remarks

We have proven in this letter that massless fields of any spin (with nonzero handedness and no gauge symmetry) in $d = 2n$-dimensional spaces, if having the Poincaré symmetry, obey the *linear equations of motion* (Eq.(1)). We have limited our proof to only even d, because the operator for handedness Eq.(6), needed in the proof, as well as the d-vector (Eq.(3)), can only be defined in even-dimensional spaces. (The generalization of the proof to all d and any signature is under consideration.)

We know that in four-dimensional space the Weyl equation is linear in the four-momentum and so are the Maxwell equations [6], describing massless spinors and massless vectors, respectively. In d-dimensional spaces the operator of handedness for spinors can be defined not only for even but also for odd dimensions. Also the Dirac-like equations exist for any dimension [5,10] and follow for even d from Eq.(1) if we take into account that for spinors $S^{ab} = -i[\gamma^a, \gamma^b]/4$:

$$(\gamma^a p_a = 0)|\Phi\rangle, \tag{28}$$

with γ^a matrices defined for d-dimensional spaces [5,10,11].

Similarly it follows from Eq.(1) [10] that vectors in d-dimensional space obey

the equations of motion

$$p_a F^{aa_1\ldots a_{n-1}} = p_a \varepsilon^{aa_1\ldots a_{n-1}}{}_{b_1 b_2\ldots b_n} F^{b_1 b_2\ldots b_n} = 0 \tag{29}$$

where $F^{abcd..}$ is a totally antisymmmetric tensor field.

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